degree of insensitivity can be achieved with a scan velocity of only 4.61 cm/s. Furthermore, the UM SD design process allows greater flexibility by generating a shaper that produces any desired scan velocity, or, conversely, any desired level of robustness.

The SD shaper contains negative impulses, so it has the possibility of exciting the higher modes of the spacecraft. If, after the dominant low mode has been eliminated by the SD shaper, the higher modes become problematic, then the above design procedure must be augmented with constraints that limit the high-mode vibration. Techniques for doing this are readily available in the literature.[15,17]

V. Conclusions

It has been shown that input shaping can be used to reduce vibration during constant-velocity scanning with flexible systems. However, the input shaping process increases the cycle time and shortens the regions over which the system travels at constant velocity. A procedure was presented to overcome these drawbacks by modifying the unshaped command before input shaping is implemented. Given cycle-time constraints, the command modification increases the scan velocity. Because this increase is proportional to the duration of the input shaper, a short-duration shaper is necessary to produce a low scan velocity. The duration of a shaper is dependent on its robustness; therefore, increasing the robustness of the input shaping process requires an increase in scan velocity. An input shaper design procedure was developed to optimally balance the trade-off between robustness and scan velocity.

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Worst-Case Distributions for Performance Evaluation of Proportionally Guided Missiles

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Nomenclature

d = gyro drift
N' = proportional navigation gain
n_C = missile's commanded acceleration
n_M = missile's acceleration
n_T = target's acceleration
R_a = line-of-sight length
t_f = conflict duration
V_C = closing velocity
V_M = missile's velocity
V_T = target's velocity
γ_M = missile's path angle
γ_T = target's path angle
λ = line-of-sight angle
τ = missile's time constant

I. Introduction

PERFORMANCE analysis of guided missiles involves uncertain events such as the engagement duration, the type of evasive maneuvers performed by the target, or the end-game environmental conditions. Under certain simplifying assumptions, the analysis can be performed analytically,[1] but more frequently, numerical Monte Carlo simulations are used for the performance analysis. A Monte Carlo simulation is a statistical sampling experiment on a model of the system. Uncertain parameters are treated as stochastic variables and are drawn from a random number generator based on assumed distributions. Each simulation run is preformed with a set of sampled variables, and the performance evaluation is based on postanalyzing a large number of simulations.

Recently, Barmish and Lagoa[2] investigated the problem of finding probability distributions that yield worst-case performance for general systems. Under some fairly mild assumptions, namely, 1) zero mean random variables (RV), 2) known support intervals, that is, ranges of change for the various RV, and 3) the probability density functions (PDF) are symmetrical and nonincreasing with respect to the absolute value of the RV, the worst-case distributions can be represented by truncated uniform distributions. A truncated uniform distribution is a uniform (rectangular) distribution defined over a subinterval of the original support interval. This is a very important result because it transforms the problem from infinite- to finite-dimensional space. Thus, for each random variable, a single parameter, namely, the width of its associated subinterval, characterizes the worst-case distribution. These parameters are to be searched for, to evaluate the worst-case performance of the system.

The purpose of the present research is to apply this result to the field of missile guidance. As already said, several uncertain parameters are always involved in the performance evaluation of guided missiles. To search for the worst-case distributions seems to be a viable approach, as opposed to arbitrarily assuming fixed PDF. To this end we focus here on proportionally guided missiles with the engagement duration as a single stochastic variable. Reference 3 extends the results to include stochastic evasive maneuvers in addition to the end-game duration, an extension that requires more advanced numerical optimization techniques. However, even in the

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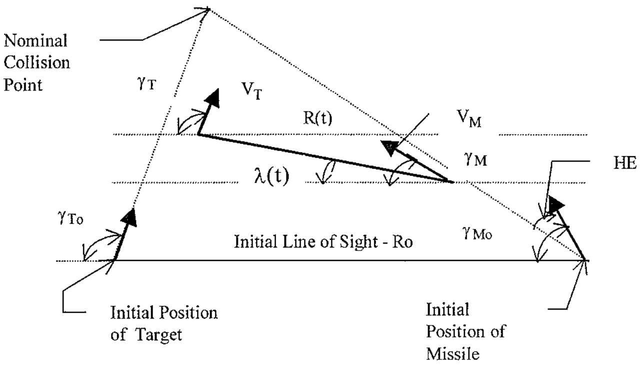

Fig. 1 Problem geometry.

single-parameter case, the solution reveals several interesting and insightful properties that are the main contributions of this Note.

This Note is organized as follows. In Sec. II, the problem is formulated as a finite-dimensional optimization problem. In Sec. III, linearized trajectories are assumed and the problem is analytically solved. Section IV presents the results of a nonlinear Monte Carlo simulation based on which the worst-case distributions are numerically found. An interesting property that enables determining the worst case by a single run is presented. Section V summarizes the results and draws some conclusions and recommendations for guidance engineers.

II. Problem Formulation

Consider a planar constant-speed end-game pursuit problem as shown in Fig. 1. Assume a proportionally guided missile with a single time lag; the equations of motion are as follows[1]:

$$\dot{R}_a = V_T \cos(\gamma_T - \lambda) - V_M \cos(\gamma_M - \lambda) = -V_C$$

$$\dot{\lambda} = [V_T \sin(\gamma_T - \lambda) - V_M \sin(\gamma_M - \lambda)]/R_a$$

$$\dot{\gamma}_T = n_T / V_T, \qquad \dot{n}_M = (n_C - n_M)/\tau$$

$$\dot{\gamma}_M = n_M \cos(\gamma_M - \lambda)/V_M, \qquad n_C = N' V_C \dot{\lambda} \tag{1}$$

We assume that the initial distance $R_a(0)$ (and, hence, the conflict duration t_f) is a stochastic variable for which the minimal and maximal values are known. We also assume the assumptions stated in the Introduction for its unknown distribution function. We will consider three different error sources: a constant target maneuver, an initial heading error (HE), and an error in the line-of-sight rate measurement (gyro drift). The missile's acceleration is typically bounded.

The success of the guided missile is measured by the obtained miss distance, which depends on the magnitude and type of the error sources, and on the end-game duration. If the statistical distribution of the conflict duration were known, then for a given error source, the expected miss distance can be obtained by averaging the resulting miss. In general, not having this distribution, we search for the worst case among the relevant truncated uniform distributions parameterized by the width $2r$ of their support interval. Thus, the problem becomes an optimization problem of finding r^* such that

$$r^* = \arg\max_r (E\{\text{miss}\}) \tag{2}$$

where r is bounded by the difference $R = (t_{f\text{-max}} - t_{f\text{-min}})/2$. Notice that all distributions are symmetrical with respect to the same average value $\mu = (t_{f\text{-max}} + t_{f\text{-min}})/2$.

III. Linear Solution

Assume that the problem can be linearized around a nominal collision course and that the missile acceleration is unbounded. If the error source is a step n_T in the target acceleration at $t = 0$, then for a specific end-game duration t_f the miss distance can be written as follows[1]:

$$\left.\frac{\text{miss}}{n_T \cdot \tau^2}\right|_{N'=3} = \frac{1}{2}\frac{t_f^2}{\tau^2}\exp\left(-\frac{t_f}{\tau}\right) \tag{3}$$

If t_f is a random variable drawn from a truncated uniform distribution of width $2r$, then we can evaluate the expected miss as follows:

$$E\left(\frac{\text{miss}}{n_T \cdot \tau^2}\right) = \int_{\mu - r}^{\mu + r} \frac{1}{2} x^2 e^{-x} \cdot \frac{1}{2r}\,\mathrm{d}x, \qquad x = \frac{t_f}{\tau} \tag{4}$$

Hence,

$$E(\text{miss}/n_T \cdot \tau^2) = e^{-\mu}\{[(\mu^2 + r^2 + 2\mu + 2)/2r]\sinh(r) - (\mu + 1)\cosh(r)\} \tag{5}$$

To find the worst-case distribution, we need to maximize Eq. (5) with respect to r.

Thus, if $r < R$ we obtain

$$\frac{\partial E(\text{miss}/n_T \cdot \tau^2)}{\partial r} = e^{-\mu}\left[\frac{\mu^2 + r^2 + 2\mu + 2}{2r}\cosh(r) \times \frac{2\mu r^2 + \mu^2 + r^2 + 2\mu + 2}{2r^2}\sinh(r)\right]\Bigg|_{r=r^*} = 0 \tag{6}$$

which is an implicit expression for r^*.

For a numerical example, assume that $\mu = 4\tau$ and $R = 4\tau$, that is, t_f may vary between 0 and 8τ. We obtain from Eq. (6) that $r^* = 2.18\tau$, and the associated expected miss is $0.15 n_T \tau^2$. Comparing this value with $0.12 n_T \tau^2$, as obtained from Eq. (4) with a uniform distribution over the entire interval $[\mu - R, \mu + R]$, we conclude that the worst-case analysis yields an expected miss that is 25% more than the naive approach.

A similar analysis can be performed for the HE, where instead of Eq. (3) we have[1]

$$\left.\frac{\text{miss}}{\text{HE} \cdot V_M \tau}\right|_{N'=3} = \frac{1}{2}\frac{t_f}{\tau}\exp\left(-\frac{t_f}{\tau}\right)\left(1 - \frac{t_f}{2\tau}\right) \tag{7}$$

Carrying out the optimization scheme (4–6), we obtain the following expression for r^*:

$$\frac{\partial E(\text{miss}/\text{HE} \cdot \tau)}{\partial r} = \frac{2}{e^2 r^2} + \frac{e^{-\mu}}{2r^2}[(\mu^2 - r^2 + 2\mu r^2)\cosh(r) - (\mu^2 r + r^3)\sinh(r)]\Bigg|_{r=r^*} = 0 \tag{8}$$

For the third error source, assuming an error d (gyro drift) in the line-of sight rate measurement, we then have[3]

$$\frac{\text{miss}}{\mathrm{d} V_C \tau^2} = \frac{1}{2}\left(\frac{t_f}{\tau}\right)^3 \exp\left[-\left(\frac{t_f}{\tau}\right)\right] \tag{9}$$

and the implicit expression for r^* becomes

$$\frac{\partial E(\text{miss}/\mathrm{d} V_C \cdot \tau^2)}{\partial r} = \frac{e^{-\mu}}{2r^2}[(3r\mu^2 + r^3 + 6\mu r + 3\mu r^3 + r\mu^3 + 6r)\cosh(r) - (6 + 3r^2\mu^2 + 3r^2\mu + 3\mu^2 + 6\mu + r^4 + \mu^3 + 3r^2)\sinh(r)]\Bigg|_{r=r^*} = 0 \tag{10}$$

For other (integer) values of N', the procedure is basically the same, and analytical expressions can still be obtained (see Ref. 3 for more examples).

IV. Nonlinear Monte Carlo Simulations

In more realistic scenarios, the nonlinear terms in Eqs. (1) can not be linearized, and the missile acceleration is bounded. To evaluate the miss distance, even for a deterministic case, numerical simulations are required. When the engagement parameters, in our case, the end-game duration, are stochastic, Monte Carlo simulations are typically used, with assumed statistical distributions for the unknown parameters. According to Ref. 3, to search for the worst-case performance, we have to go through an iterative optimization process where each iteration is related to a particular $2r$-width uniform distribution. Each iteration, therefore, involves a set of n Monte Carlo runs. Naturally, reducing the required number of runs is of great practical importance.

For a numerical example we consider a head-on conflict, with the following parameters: $N' = 3$, $\tau = 0.5$ s, $V_M = 700$ m/s, $V_T = 300$ m/s, and $V_C = 1000$ m/s. We further assume a constant target maneuver at $t = 0$ of $n_T = 50$ m/s^2. The conflict duration is our only RV, and we will continue with the assumption that $\mu = 4\tau$ and $R = 4\tau$, that is, t_f varies between 0 and 4 s with a mean value of 2 s, corresponding to $R_a(0)$ variations between 0 and 4 km.

We consider the case with unbounded missile acceleration. Recall that for this case the linear analysis yielded $r^* = 2.18\tau$. Figure 2 presents the average miss distance for a different number of runs as a function of r, the width of the underlying uniform distribution. Notice that the worst-case r value can be identified (approximately 2.2) even for the case of $n = 50$ runs, although the miss distance has not converged yet. This phenomenon repeats itself for other error sources[3] and can be useful when more variables are considered and when optimization techniques that require many function evaluations are employed. In our one-dimensional problem, the worst case can be obtained by a simple search; hence, the amount of runs is of less significance.

Figure 3 presents the normalized miss distance for different bounds on the missile actual acceleration. A very interesting phenomenon is revealed. As the bound on the missile acceleration gets tighter, the truncated uniform distribution shrinks into a delta function of zero width. Notice that these more limited-missile cases are realistic in terms of missile-to-target acceleration ratios. Thus, the worst-case miss is obtained by a single run with the mean value for the conflict duration.

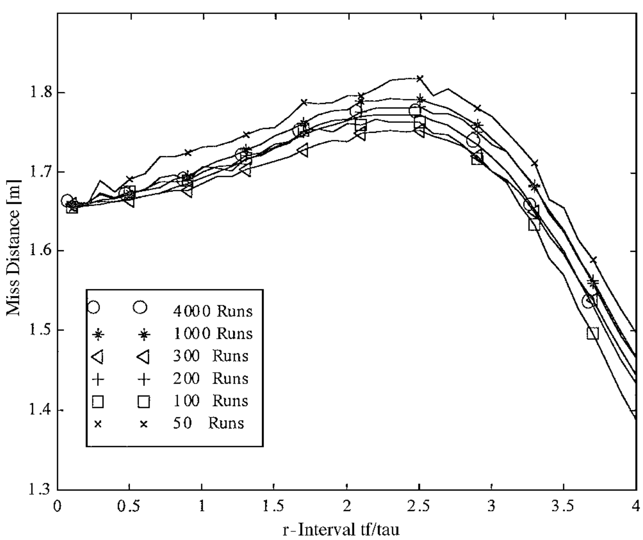

Fig. 2 Monte Carlo results for different number of runs.

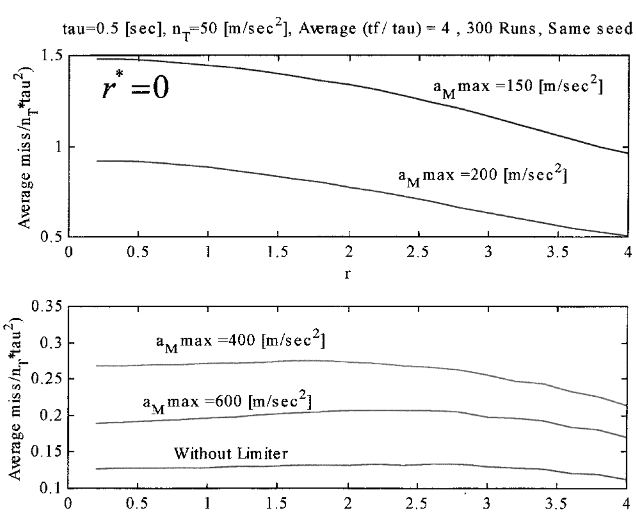

Fig. 3 Monte Carlo results for various acceleration bounds.

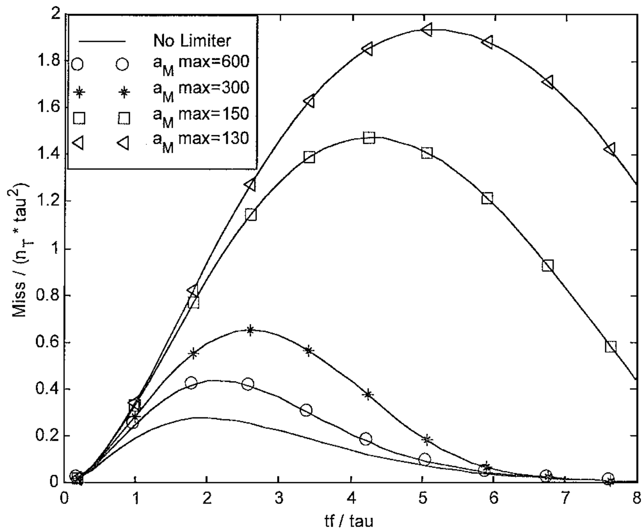

Fig. 4 Miss distance function for various acceleration bounds.

To understand this result, we consider the results of deterministic simulations in terms of the obtained miss distance as a function of the conflict duration as shown in Fig. 4. This function becomes concave as we lower the bound on the missile acceleration. The described phenomenon is a result of the following theorem for concave functions.

Theorem: Let f be a concave function on $[\mu - R, \mu + R]$, then

$$E\{f(x)\} = \int_{\mu - r}^{\mu + r} f(x) \cdot \frac{1}{2r}\,\mathrm{d}x \le f(\mu), \qquad \forall r \le R \tag{11}$$

Proof: By concavity we have

$$f[\alpha x + (1-\alpha)x'] \ge \alpha f(x) + (1-\alpha) f(x'), \qquad \forall 0 < \alpha < 1$$

$$x, x' \in \text{domain of } f \tag{12}$$

Let $x = \mu - h$, $x' = \mu + h$, and $\alpha = 0.5$, then

$$f(\mu) \ge \tfrac{1}{2}[f(\mu - h) + f(\mu + h)] \tag{13}$$

By integrating both sides, we obtain after some simple manipulations

$$\frac{1}{r}\int_0^r f(\mu)\,\mathrm{d}h \ge \frac{1}{r}\int_0^r \frac{1}{2}[f(\mu - h) + f(\mu + h)]\,\mathrm{d}h$$

$$= \int_{\mu - r}^{\mu + r} f(y)\frac{1}{2r}\,\mathrm{d}y \tag{14}$$

Because the left-hand side is equal to $f(\mu)$, the inequality of Eq. (11) is established.

We conclude that whenever the function f, the miss distance in our problem, is concave, the evaluation at the mean provides the worst-case performance in the earlier-defined statistical sense.

Remark: Reference 3 establishes a similar theorem for convex functions. It is shown that if convexity is valid over the uncertainty interval, then we get $r^* = R$ for the worst-case distribution.[3]

V. Conclusions

Analytical and numerical methods to assess worst-case performance of proportionally guided missiles were studied. According to a recently published theory of Ref. 2, the worst-case probability distribution, that which yields the minimal “probability of performance" satisfaction, is characterized as a truncated uniform distribution. This result was applied to the performance analysis of engagement duration and of several error sources. Linear theory enables the formulation of analytical solutions to the problem. When numerical Monte Carlo simulations are used to search for the worst-case distribution, a limited number of simulation runs typically are needed for the search. Moreover, when the miss distance is a concave function of the conflict duration on a certain interval, the worst-case miss distance can be determined from a singe Monte Carlo trial.

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Equations for Approximate Solutions Using Variational Calculus

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Introduction

TO develop differential calculus, neighboring points are used to derive the formulas for the derivatives of particular functions. Once these formulas are known, differentiation of complicated functions can be performed directly. There are many problems involving neighboring curves, for example, applications of regular perturbation theory, where Taylor series expansions are used to derive the formulas for variations of particular functionals. However, the step to taking variations has not been made. The purpose of this Note is to call attention to variational calculus.[1] Although the concepts of this Note apply to all types of equations (algebraic, differential, integral, etc.) and all types of problems (initial value, boundary value, etc.), the discussion is limited to initial-value problems involving nonlinear ordinary differential equations.

There are many avenues to the creation of an approximate analytical solution. Consider a single differential equation. 1) Approximations can be introduced into the differential equation and small terms simply discarded. If the resulting solution is accurate enough, work stops. If not, something more is needed. 2) Another possibility is to use error compensation,[2,3] that is, to replace the small terms in method 1 by a small constant to compensate for the errors. Then, by the using of the methods of regular perturbation theory,[4] a correction term can be found for the so-called zeroth-order solution in method 1. By the adjusting of the size of the small constant, the magnitude of the correction can be changed so that the approximate solution better fits the exact solution. 3) Another method is to find a small parameter in the differential equation and to use regular perturbation theory. This method has been used to obtain an approximate optimal control for the aeroassisted orbital plane change problem.[5] 4) Still another method is to assume that the solution lies in the neighborhood of a nominal solution. This is the approach that is used to obtain the Clohessy–Wiltshire equations for studying the motion of a mass relative to an orbiting mass.[6] Actually, methods 3 and 4 are similar in form; in method 3, the perturbed solution is caused by a perturbed parameter, whereas in method 4, the perturbed solution is caused by a perturbed initial condition.

Regardless of the method used, the standard procedure is to use Taylor series expansions to obtain the equations to be solved for the approximate solution. However, the expansion approach can be very involved, and often it is difficult to ensure that the resulting equations are correct. The messages of this Note are that the expansion approach and the variational approach[1] are equivalent, that the variational approach is simpler, and that it is helpful to have more than one way to derive the equations for the approximate solution to be sure they are correct.

In what follows, the expansion approach is reviewed briefly, the variational approach is discussed, and a satellite problem involving a small parameter is discussed to compare the two approaches. Finally, the Clohessy–Wiltshire equations are derived using variations.

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Differential Equation and Initial Conditions

A problem in the realm of regular perturbation theory is to obtain an approximate analytical solution of a nonlinear ordinary differential equation having the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \varepsilon \boldsymbol{g}(\boldsymbol{x}) \tag{1}$$

where $\boldsymbol{x}$ is an $n \times 1$ vector, the dot denotes a derivative with respect to time, and the scalar ε denotes a small parameter. The solution of the differential equation must satisfy the initial conditions

$$t_i = t_{i_s}, \qquad \boldsymbol{x}_i = \boldsymbol{x}_{i_s} \tag{2}$$

where the subscript s denotes a specific value.

It is assumed that a solution of the differential equation subject to the initial conditions exists. The functional form of the solution is given by

$$\boldsymbol{x} = \boldsymbol{x}(t, \varepsilon, t_i, \boldsymbol{x}_i) \tag{3}$$

For the case where the initial conditions are not to be changed, the solution can be rewritten in the form

$$\boldsymbol{x} = \boldsymbol{x}(t, \varepsilon) \tag{4}$$

Taylor Series Approach

Expanding Eq. (4) in terms of the small parameter ε leads to

$$\boldsymbol{x} = \boldsymbol{x}(t, 0) + \boldsymbol{x}_\varepsilon(t, 0)\varepsilon + \tfrac{1}{2!}\boldsymbol{x}_{\varepsilon\varepsilon}(t, 0)\varepsilon^2 + \cdots \tag{5}$$

where the subscript ε denotes a partial derivative with respect to ε. Hence, the solution of Eq. (1) can be assumed to have the form

$$\boldsymbol{x} = \boldsymbol{x}_0(t) + \boldsymbol{x}_1(t)\varepsilon + \boldsymbol{x}_2(t)\varepsilon^2 + \cdots \tag{6}$$

In practice, the Taylor series approach is to substitute Eq. (6) into a particular differential equation and initial conditions, carry out whatever mathematical operations and expansions are needed to write the equations in terms of powers of ε, and equate the coefficient of each power of ε to zero to obtain the differential equations and initial conditions for $x_0, x_1, x_2, \ldots$.

To illustrate the Taylor series approach, consider a satellite moving in the equatorial plane of an oblate spheroid Earth. The motion of the satellite is governed by the equation[7]

$$\ddot{r} - h^2/r^3 = -(\mu/r^2)[1 + (R^2/r^2)\varepsilon] \tag{7}$$

where r is the radial distance to the satellite, h is the constant angular momentum of the satellite, R is the radius of the Earth, μ is the gravitational constant, and $\varepsilon = (3/2)J_2 = 0.001624$ is the flatness. The initial conditions are given by

$$t_i = t_{i_s}, \qquad r_i = r_{i_s}, \qquad \dot{r}_i = \dot{r}_{i_s} \tag{8}$$